# Problem Solving through problems

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# 1 Heuristics

**Problem. 1.1.6** Beginning with 2 and 7, the sequence $2, 7, 1, 4, 2, 8, ..$ is constructed by multiplying successive pairs of its members and adjoining the result as the next one or two numbers of the sequence, depending on whether the product is a one- or a two-digit number. Prove that the digit 6 appears an infinite number of times in the sequence.

*Solution.* □

**Problem. 1.1.12** Let $S$ be a set, and let $*$ be a binary operation on $S$ satisfying the laws

$$x * (x * y) = y \text{ for all } x, y \text{ in } S$$
$$(y * x) * x = y \text{ for all } x, y \text{ in } S$$

*Solution.* We have

□

## §1.1 Choose effective notation

**Problem. 1.5.5** Write an equation to represent the following statements:

(a) At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel.

(b) There are six times as many students as professors at this college.

*Solution.* (a) Let $c$ be the number of people ordering cheesecake. Let $s$ be the number of people ordering strudel. We have $\frac{c}{s} = \frac{4}{5}$.

(b) Let $s$ be the number of students. Let $p$ be the number of professors. We have $s = 6p$.

□

**Problem. 1.5.6** Guy wires are strung from the top of each of two poles to the base of the other.What is the height from the ground where the two wires cross?

*Solution.* Let $a$ and $b$ be the heights of the two poles. Let $x$ be the horizontal distance from the base of the first pole till the projection of the point (where the two wires cross) on the ground.Let $y$ be this distance for the other pole. Let $h$ be the height from the ground where the two wires cross. Using similar triangles, we have

$$\frac{h}{x} = \frac{b}{x + y}$$
$$\frac{h}{y} = \frac{a}{x + y}$$

From the above we see that $h = \frac{ab}{a+b}$ □

**Problem. 1.5.7** A piece of paper 8 inches wide is folded so that one corner is placed on the opposite side.Express the length of crease, L, in terms of the angle θ alone.

*Solution.* We have

$$L\cos\theta\sin 2\theta = 8$$
$$\implies L = \frac{4}{\sin\theta\cos^2\theta}$$

□

**Problem. 1.5.8** Let $P_1, P_2, \ldots, P_{12}$ be the successive vertices of a regular dodecagon(twelve sides).Are the diagonals $P_1P_9$, $P_2P_11$, $P_4P_12$ concurrent?

*Solution.* □

**Problem. 1.6.11** The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is always a perfect square.Verify this identity by incorporating symmetry into the notation.

*Solution.* Let $a-3d, a-d, a+d, a+3d$ be four consecutive terms of an arithmetic progression with common difference $2d$. We have,

$$\begin{aligned}(a-3d)(a-d)(a+d)(a+3d)+16d^4\\ =(a^2-9d^2)(a^2-d^2)+16d^4\\ =a^4-10a^2d^2+25d^4\\ =(a^2-5d^2)^2\end{aligned}$$

□

## §1.2 Argue By Contradiction

## §1.3 Pursue Parity

**Problem. 1.10.6**

(a) Remove the lower left corner square and the upper right corner square from an ordinary $8-by-8$ chessboard.Can the resulting board be covered by 31 dominos.

(b) Let thirteen points $P1, \ldots, P_{13}$ be given in the plane, and suppose they are connected by the segments $P_1P_2, P_2P_3, \ldots, P_{12}P_{13}, P_{13}P_1$. Is it possible to draw a straight line which passes through the interior of each of these segments?

*Solution.* (a) The two corner squares are of the same colour. After removing the two corner square we either have 30 white squares or 30 black squares. Each domino covers one white square and one black squares, so 31 dominos cover 31 white squares and 31 black squares.Therefore, 31 dominoes cannot cover the chessboard from which the corner squares have been removed.

(b)

$\square$

**Problem. 1.10.8** Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2 \ldots, n$. Prove that $n$ is odd, the product

$$(a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$$

is an even number.

*Solution.* As $n$ is odd, you have one additional $a_i$ which is odd compared to number of even $a_i$s. Therefore the above product has at least one even number. $\square$

**Problem. 1.10.10** Show that $x^2 - y^2 = a^3$ always has integral solutions for $x$ and $y$ whenever $a$ is a positive integer.

*Solution.* We have two cases:

1. a = 2k, where $k$ is an integer.
   From $x^2 - y^2 = 8k^3$, We get the following equations
   $$x + y = 4k^2$$
   $$x - y = 2k$$
   we can see that
   $$x = k + 2k^2$$
   $$y = 2k^2 - k$$
   which are both integers.

2. a = 2k + 1, where $k$ is an integer
   From $x^2 - y^2 = (2k+1)^3$, we get the following equations
   $$x + y = 4k^2 + 4k + 1$$
   $$x - y = 2k + 1$$
   we can see that
   $$x = 2k^2 + 3k + 1$$
   $$y = 2k^2 + k$$
   which are both integers.

$\square$

## §1.4 Generalize

**Problem. 1.12.4** By setting $x$ equal to the appropriate values in the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

(or one of its derivatives, etc.) evaluate each of the following:

(a) $\sum_{k=1}^{n} k^2 \binom{n}{k}$

(b) $\sum_{k=1}^{n} 3^k \binom{n}{k}$

(c) $\sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k}$

(d) $\sum_{k=1}^{n} (2k+1) \binom{n}{k}$

*Solution.* We start with the binomial expansion

(a) Differentiating the binomial expansion once, we have

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

Multiplying the above by $x$ and differentiating again, we have

$$n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1} = \sum_{k=1}^{n} k^2 \binom{n}{k} x^{k-1}$$

Setting $x = 1$ in the above equation we have

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1} = n(n+1)2^{n-2}$$

(b) Setting $x = 3$ in the binomial expansion we have

$$\sum_{k=1}^{n} 3^k \binom{n}{k} = 4^n$$

(c) Integrating the binomial expansion we have

$$\int_0^1 (1+x)^n dx = \frac{2^{n+1}-1}{n+1} = 1 + \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k}$$
$$\implies \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-n-2}{n+1}$$

(d) We have

$$\sum_{k=1}^{n} (2k+1) \binom{n}{k} = 2n2^{n-1} + 2^n - 1 = (n+1)2^n - 1$$

□

**Problem. 1.12.5** Evaluate

$$\det \begin{bmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{bmatrix}$$

*Solution.* Replacing $d$ by $x$ in the last row, we get a polynomial $P(x)$.

$$P(x) = \det \begin{bmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & x & x^2 & x^4 \end{bmatrix}$$

$P(x)$ is a polynomial of degree 4. Moreover, $P(a) = 0$, $P(b) = 0$ and $P(c) = 0$, since the corresponding matrix, with $d$ replaced by $a$ or $b$ or $c$ respectively, then has two identical rows. Therefore

$$P(x) = A(x-a)(x-b)(x-c)(x-l)$$

where $l$ is the fourth root of $P(x) = 0$.

The coefficient of $x^3$ in $P(x) = 0$ is zero,so the sum of the roots $a+b+c+l = 0$. Therefore $l = -(a+b+c)$. The coefficient of $x^4$ is

$$A = \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

Therefore, the value of the original determinant is given by

$$P(d) = (b-a)(c-a)(c-b)(d-a)(d-b)(d-c)(a+b+c+d)$$

□

**Problem. 1.12.6**

(a) Evaluate $\int_0^\infty (e^{-x}\sin x)/x dx$.

(b) Evaluate $\int_0^1 (x-1)/\ln x dx$.

(c) Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$$

*Solution.* We make use of differentiation under integral sign or parameter differentiation.

(a) Using parameter differentiation,

$$\begin{aligned} G(k) &= \int_0^\infty (e^{-x}\sin(kx))/x dx \\ \implies \frac{dG(k)}{dk} &= \int_0^\infty e^{-x}\cos kx dx \end{aligned}$$

We have

$$\begin{aligned} \int_0^\infty e^{-x}\cos(kx)dx &= -e^{-x}\cos(kx)\Big|_0^\infty - k\int_0^\infty e^{-x}\sin(kx)dx \\ &= 1 - k\left(-e^{-x}\sin(kx)\Big|_0^\infty + k\int_0^\infty e^{-x}\cos(kx)dx\right) \\ &= 1 - k^2\int_0^\infty e^{-x}\cos(kx)dx \\ \implies \int_0^\infty e^{-x}\cos(kx)dx &= \frac{1}{k^2+1} \end{aligned}$$

Hence,

$$\frac{dG(k)}{dk} = \int_0^\infty e^{-x}\cos(kx)dx = \frac{1}{k^2+1}$$

The solution of the differential equation with $G(0) = 0$ is $G(k) = \arctan(k)$.

Therefore,

$$\int_0^\infty (e^{-x}\sin x)/x dx = G(1) = \arctan(1) = \frac{\pi}{4}$$

(b) Using parameter differentiation,

$$H(m) = \int_0^1 (x^m - 1)/\ln x dx$$
$$\implies \frac{dH(m)}{dm} = \int_0^1 x^m dx = \frac{1}{m+1}$$

The solution of the differential equation with $H(0) = 0$ is $H(m) = \ln(m+1)$.

Therefore,

$$\int_0^1 (x-1)/\ln x dx = H(1) = \ln(2)$$

(c) Using parameter differentiation,

$$F(a) = \int_0^\infty \frac{\arctan(ax) - \arctan(x)}{x} dx$$
$$\implies \frac{dF(a)}{da} = \int_0^\infty \frac{1}{a^2x^2+1} dx = \frac{\pi}{2a}$$

The solution of the differential equation with $F(1) = 0$ is $F(a) = \frac{\pi}{2}\ln(a)$.

Therefore,

$$\int_0^\infty \frac{\arctan(ax) - \arctan(x)}{x} dx = F(\pi) = \frac{\pi}{2}\ln(\pi)$$

□

**Problem. 1.12.7** Which is larger $\sqrt[3]{60}$ or $2 + \sqrt[3]{7}$?

*Solution.* Let $x = a^3$ and $y = b^3$. We have

$$(\sqrt[3]{4(x+y)})^3 = 4(x+y) = 4(a^3+b^3) \tag{1.1}$$
$$(\sqrt[3]{x} + \sqrt[3]{y})^3 = x + y + 3\sqrt[3]{xy}(\sqrt[3]{x} + \sqrt[3]{y}) = a^3 + b^3 + 3ab(a+b) \tag{1.2}$$

Subtracting 1.2 from 1.1, we have

$$4(a^3+b^3) - (a^3+b^3+3a^2b+3ab^2) = 3(a^3+b^3-a^2b-ab^2) \geqslant 0$$

because of Muirhead's inequality as $(3,0)$ majorizes $(2,1)$. Equality holds only when $a = b$.

Therefore,

$$\sqrt[3]{60} = \sqrt[3]{4(8+7))^3} > \sqrt[3]{8} + \sqrt[3]{7}$$

□

# 2 Two Important Principles:Induction and Pigeonhole

## §2.1 Induction: Build on P(k)

## §2.2 Pigeonhole Principle

# 3 Arithmetic

## §3.1 Modular Arithmetic

**Problem. 3.2.11** Prove that any subset of 55 numbers chosen from the set $\{1, 2, \ldots, 100\}$ must contain numbers differing by $10, 12$ and $13$, but need not contain a pair differing by $11$.

**Problem. 3.2.12** The elements of a determinant are arbitrary integers.Determine the probability that the value of the determinant is odd.

**Problem. 3.2.13**

(a) Determine whether the following matrix is singular or nonsingular:

$$\begin{bmatrix} 54401 & 57668 & 15982 & 103790 \\ 33223 & 26563 & 23165 & 71489 \\ 36799 & 37189 & 16596 & 46152 \\ 21689 & 55538 & 79922 & 51237 \end{bmatrix}$$

(b) Determine whether the following matrix is singular or nonsingular:

$$\begin{bmatrix} 64809 & 99185 & 42391 & 44350 \\ 61372 & 26563 & 23165 & 71489 \\ 82561 & 39189 & 16596 & 46152 \\ 39177 & 55538 & 79922 & 51237 \end{bmatrix}$$

**Problem. 3.2.14**

(a) Show that $2^{2x+1} + 1$ is divisible by 3.

(b) Prove or disprove:$2^x \equiv 2^y (\mathrm{mod} n)$ if $x \equiv y (\mathrm{mod} n)$.

(c) Show that $4^{3x+1} + 2^{3x+1} + 1$ is divisible by 7.

(d) If $n > 0$, prove that 12 divides $n^4 - 4n^3 + 5n^2 - 2n$.

(e) Prove that $2903^n - 803^n - 464 * n + 261^n$ is divisible by 1897.

**Problem. 3.2.15**

(a) Prove that no prime three more than a multiple of four is a sum of two squares.

(b) Prove that the sequence (in base-10 notation)

$$11, 111, 1111, 11111, \ldots$$

(c) Prove that the difference of the squares of any two odd numbers is exactly divisible by 8.

(d) Prove that $2^{70} + 3^{70}$ is divisible by 13.

(e) Prove that the sum of two odd squares cannot be a square.

(f) Determine all integral solutions of $a^2 + b^2 + c^2 = a^2b^2$.

**Problem. 3.2.16**

(a) If $x^3 + y^3 = z^3$ has a solution in integers $x, y, z$, show that one of the three must be a multiple of 7.

(b) If $n$ is positive integer greater than 1 such that $2^n + n^2$ is prime, show that $n \equiv 3(\mod 6)$.

(c) Let $x$ be an integer one less than a multiple of 24.Prove that if $a$ and $b$ are positive integers such that $ab = x$, then $a + b$ is a multiple of 24.

(d) Prove that if $n^2 + m$ and $n^2 - m$ are perfect squares, then $m$ is divisible by 24.

**Problem. 3.2.17** Let $S$ be a set of primes such that $a, b \in S$($a$ and $b$ need not be distinct) implies $ab+4 \in S$. Show that $S$ must be empty.

**Problem. 3.2.18** Prove that there are no integers $x$ and $y$ for which

$$x^2 + 3xy - 2y^2 = 122.$$

**Problem. 3.2.19** Given an integer $n$, show that an integer can always be found which contains only the digits 0 and 1 and which is divisible by $n$.

**Problem. 3.2.20** Show that if $n$ divides a single Fibonacci number, then it will divide infinitely many Fibonacci numbers.

**Problem. 3.2.21** Suppose that $a$ and $n$ are integers, $n > 1$.Prove that the equation $ax \equiv 1(\mod n)$ has a solution if and only if $a$ and $n$ are relatively prime.

**Problem. 3.2.22** Let $a, b, c, d$ be fixed integers with $d$ not divisible by 5. Assume that $m$ is an integer for which

$$am^3 + bm^2 + cm + d$$

is divisible by 5. Prove that there exists an integer $n$ for which

$$dn^3 + cn^2 + bn + a$$

is also divisible by 5.

**Problem. 3.2.23** Prove that $(21n - 3)/4$ and $(15n + 2)/4$ cannot both be integers for the same positive integer $n$.

**Problem. 3.2.25** Let $m_0, m_1, \ldots, m_r$ be positive integers which are piarwise relatively prime. Show that there exist $r + 1$ consecutive integers $s, s + 1, \ldots, s + r$ such that $m_i$ divides $s + i$ for $i = 0, 1, \ldots, r$.

## §3.2 Positional Notation

**Problem. 3.4.7** Prove that there does not exist an integer which is doubled when the initial digit is transferred to the end.

**Problem. 3.4.10** Given a two-pan balance and a system of weights of $1, 3, 3^2, 3^3, \dots$ pounds, show that one can weigh any integral number of pounds (weights can be put into either pan).

*Solution.* □

**Problem. 3.4.11**

(a) Does the number $0.1234567891011121314\dots$, which is obtained by writing successively all the integers, represent a rational number?

(b) Does the number $0.011010100010100\dots$, where $a_=1$ if $n$ is prime, 0 otherwise, represent a rational number?

**Problem. 3.4.12** Let $S = a_0a_1a_2\dots$, where $a_n = 0$ if there are an even number of $1's$ in the expression of $n$ in base 2 and $a_n = 1$ if there are an odd number of $1's$. Thus, $S = 01101001100\dots$ Define $T = b_1b_2b_3\dots$ wher $b_i$ is the number of $1's$ between the $ith$ and the $(i+1)st$ occurence of 0 in $S$. Thus, $T = 2102012\dots$ Prove that $T$ contains only three symbols $0, 1, 2$.

**Problem. 3.4.13** Show that there is a one-to-one correspondence between the points of the closed interval $[0, 1]$ and the points of the open interval $(0, 1)$. Give an explicit description of such a correspondence.

## §3.3 Arithmetic of Complex Numbers

**Problem. 3.5.6**

(a) Given that $13 = 2^2 + 3^2$ and $74 = 5^2 + 7^2$, express $13 \times 74 = 962$ as a sum of two squares.

(b) Show that $4arctan\frac{1}{5} - arctan\frac{1}{239} = \frac{\pi}{4}$.

*Solution.* Let $z = 2 + 3i, w = 5 + 7i$.We have

$$13 \times 74 = |z|^2|w|^2 = |zw|^2 = |-11 + 29i| = 11^2 + 29^2$$

□

**Problem. 3.5.7** Suppose $A$ is a complex number and $n$ is a positive integer such that $A^n = 1$ and $(A+1)^n = 1$. Prove that $n$ is divisible by 6 and that $A^3 = 1$

**Problem. 3.5.8** Show that

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \dots = 2^{n/2}cos\frac{n\pi}{4}$$

and

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \dots = 2^{n/2}sin\frac{n\pi}{4}$$

*Solution.* We have

$$(1+i)^n = \binom{n}{0} + i\binom{n}{1} - \binom{n}{2} - i\binom{n}{3} + \cdots$$
$$(1-i)^n = \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \cdots$$

Therefore,

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = \frac{(1+i)^n + (1-i)^n}{2} = 2^{n/2}\frac{e^{in\pi/4} + e^{-in\pi/4}}{2} = 2^{n/2}\cos\frac{n\pi}{4}$$
$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = \frac{(1+i)^n - (1-i)^n}{2} = 2^{n/2}\frac{e^{in\pi/4} - e^{-in\pi/4}}{2} = 2^{n/2}\sin\frac{n\pi}{4}$$

□

**Problem.** **3.5.9** By considering possible magnitudes and arguments,

(a) find all values of $\sqrt[3]{-i}$;

(b) find which values of $(3-4i)^{-3/8}$ lie closest to the imaginary axis

*Solution.* (a) We have $-i = e^{2k\pi - \pi/2}$.Therefore,

$$\sqrt[3]{-i} = e^{2k\pi/3 - \pi/6}$$

where $k = 0, 1, 2$.

(b)

□

**Problem.** **3.5.10**

(a) Prove that if $x - x^{-1} = 2i\sin\theta$ then $x^n - x^{-n} = 2i\sin(n\theta)$.

(b) Using part(a), express $\sin^{2n}\theta$ as a sum of sines whose angles are multiple of $\theta$.

*Solution.* (a)

(b)

□

**Problem.** **3.5.11** Show that

$$\tan(n\theta) = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \cdots}{\binom{n}{0} - \binom{n}{2}\tan^2\theta + \cdots}$$

*Solution.* □

**Problem.** **3.5.14** Show that if $e^{i\theta}$ satisfies the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$, where the $a_i$ are real, then $a_{n-1}\sin\theta + a_{n-2}\sin 2\theta + \cdots + a_1\sin(n-1)\theta + a_0\sin(n\theta) = 0$.

*Solution.* If $e^{i\theta}$ is a solution of the equation, then $e^{-i\theta}$ is also a solution of the equation as the coefficients are real.We have

$$a_n + a_{n-1}e^{i\theta} + \cdots + a_0 e^{in\theta} = 0$$

Equating the imaginary part of the above equation to zero, we get

$$a_{n-1} sin\theta + a_{n-2} sin2\theta + \cdots + a_1 sin(n-1)\theta + a_0 sin(n\theta) = 0$$

□

# 4 Algebra

## §4.1 Alegbraic Identities

**Problem. 4.1.5**

(a) If $a$ and $b$ are consecutive integers, show that $a^2 + b^2 + (ab)^2$ is a perfect square.

(b) If $2a$ is the harmonic mean of $b$ and $c$, show that the sum of the squares of the three number $a, b,$ and $c$ is the square of a rational number.

(c) If $N$ differs from two successive squares between which it lies by $x$ and $y$ respectively, prove that $N - xy$ is a square.

*Solution.* (a) If $a = n$ and $b = n + 1$, we have

$$\begin{aligned} a^2 + b^2 + (ab)^2 &= n^2 + (n+1)^2 + n^2(n+1)^2 \\ &= n^4 + 2n^3 + 3n^2 + 2n + 1 \\ &= (n^2 + n + 1)^2 \end{aligned}$$

(b) We have

$$\begin{aligned} a^2 + b^2 + c^2 &= \frac{b^2c^2}{(b+c)^2} + b^2 + c^2 \\ &= \frac{b^4 + c^4 + 3b^2c^2 + 2b^3c + 2c^3b}{(b+c)^2} \\ &= \left(\frac{b^2 + c^2 + bc}{b+c}\right)^2 \end{aligned}$$

(c) Let $x = N - n^2$ and $y = (n+1)^2 - N$, we have

$$\begin{aligned} N - xy &= N - (N - n^2)((n+1)^2 - N) \\ &= n^2(n+1)^2 + N^2 + N(1 - (n+1)^2 - n^2) \\ &= (N - n(n+1))^2 \end{aligned}$$

□

**Problem. 4.1.6** Prove that there are infinitely many natural numbers $a$ with the following property: The number $n^4 + a$ is not prime for any natural number $n$.

**Problem. 4.1.7** Suppose that an integer $n$ is the sum of two triangular numbers,

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2}$$

write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how $x$ and $y$ can be expressed in terms of $a$ and $b$.

Show that conversely, if $4n + 1 = x^2 + y^2$, the $n$ is the sum of two triangular numbers.

*Solution.* We have,

$$\begin{aligned} 4n + 1 &= 2(a^2 + a) + 2(b^2 + b) + 1 \\ &= (a + b + 1)^2 + (a - b)^2 \end{aligned}$$

If $4n + 1 = x^2 + y^2$, we have

$$\begin{aligned} n &= \frac{x^2 + y^2 - 1}{4} \\ &= \frac{(x + y - 1)}{2}\frac{(x + y + 1)}{2}/2 + \frac{(x - y - 1)}{2}\frac{(x - y + 1)}{2}/2 \end{aligned}$$

□

**Problem. 4.1.8** Let N be the number which when expressed in decimal notation consists of 91 ones:

$$N = \underbrace{111\ldots1}_{91}.$$

Show that N is a composite number.

*Solution.* We have

$$N = \frac{10^{91} - 1}{10 - 1} = \frac{(10^7)^{13} - 1}{10 - 1}$$

As $a^n - b^n$ is divisible by $a - b$, the numerator is divisible by $10^7 - 1 = 9 \cdot 1111111$. As N is divisible by 1111111, N is a composite number. □

**Problem. 4.1.9** Prove that any two numbers in the following sequence are relatively prime:

$$2 + 1, 2^2 + 1, 2^4 + 1, 2^8 + 1, \ldots, 2^{2^n} + 1, \ldots$$

Show that this result proves that there are infinite number of primes.

*Solution.* □

**Problem. 4.1.10** Determine all triplets of integers satisfying the equation:

$$x^3 + y^3 + z^3 = (x + y + z)^3$$

*Solution.* We have

$$\begin{aligned} (x^3 + y^3 + z^3) + 3(x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2) + 6xyz &= (x + y + z)^3 \\ \implies 3\left((x + y + z)(xy + xz + yz) - 3xyz\right) + 6xyz &= 0 \\ \implies (x + y + z)(xy + xz + yz) &= xyz \end{aligned}$$

□

## §4.2 Unique Factorization of Polynomials

**Problem. 4.2.7** Find polynomials $F(x)$ and $G(x)$ such that

$$(x^8 - 1)F(x) + (x^5 - 1)G(x) = x - 1$$

**Problem. 4.2.8** What is the greatest common divisor of $x^n - 1$ and $x^m - 1$?

*Solution.* The greatest common divisor of $x^n - 1$ and $x^m - 1$ is $x^{gcd(m,n)} - 1$ as $x^n - 1$ is divisible $x^k - 1$ whenever $k$ divides $n$. □

**Problem. 4.2.9** Let $f(x)$ be a polynomial leaving a remained $A$ when divided by $x - a$ and the remainder $B$ when divided $x - b$, $a \neq b$. Find the remainder when $f(x)$ is divided by $(x - a)(x - b)$.

*Solution.* We have

$$\begin{aligned} f(x) &= q(x)(x-a)(x-b) + cx + d \\ \implies f(a) &= A = ca + d \wedge f(b) = B = cb + d \end{aligned}$$

Solving for $c$ and $d$ we get the remainder when $f(x)$ is divided by $(x - a)(x - b)$ as

$$\frac{A - B}{a - b}x + \frac{bA - aB}{b - a}$$

□

**Problem. 4.2.10** Show that $x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$,a,b,c,d positive integers is divisible by $x^3 + x^2 + x + 1$.

*Solution.* Let $f(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$. We have $f(-1) = 0, f(i) = 0$ and $f(-i) = 0$, therefore by the Factor theorem,we see that $(x + 1)$, $(x - i)$ and $(x + i)$ are factors of $f(x)$. Therefore $f(x)$ is divisible by $(x + 1)(x - i)(x + i) = 1 + x + x^2 + x^3$. □

**Problem. 4.2.11** Show that the polynomials $(\cos\theta + x\sin\theta)^n - \cos(n\theta) - x\sin(n\theta)$ is divisible by $x^2 + 1$.

*Solution.* Let $f(x) = (\cos\theta + x\sin\theta)^n - \cos(n\theta) - x\sin(n\theta)$.We have $f(i) = 0$ and $f(-i) = 0$ using De Moivre's Theorem. Using the Factor theorem,we get that $(x + i)$ and $(x - i)$ are factors of $f(x)$. Therefore $f(x)$ is divisible by $(x - i)(x + i) = 1 + x^2$. □

**Problem. 4.2.12** For what $n$ is the polynomial $1 + x^2 + x^4 + \cdots + x^{2n-2}$ divisible by the polynomial $1 + x^2 + x^3 + \cdots + x^{n-1}$.

*Solution.* We have

$$\begin{aligned} 1 + x^2 + x^4 + \cdots + x^{2n-2} &= \frac{x^{2n} - 1}{x^2 - 1} \\ 1 + x^2 + x^3 + \cdots + x^{n-1} &= \frac{x^{2n} - 1}{x^2 - 1} \end{aligned}$$

Dividing the first expression by the second we get

$$\frac{(x^{2n} - 1)(x - 1)}{(x^2 - 1)(x^n - 1)} = \frac{x^n + 1}{x + 1}$$

$x^n + 1$ is divisible by $x + 1$ when $n$ is odd, therefore the polynomial $1 + x^2 + x^4 + \cdots + x^{2n-2}$ divisible by the polynomial $1 + x^2 + x^3 + \cdots + x^{n-1}$ when $n$ is odd. □



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**Problem. 4.2.13** A real number is called *algebraic* if it is a zero of a polynomial with integer coefficients.

(a) Show that $\sqrt{2}+\sqrt{3}$ is algebraic.

(b) Show that $\cos(\pi/2n)$ is algebraic for each positive integer $n$.

*Solution.* (a) We have

$$\begin{aligned} x-\sqrt{2}-\sqrt{3}&=0\\ \implies x^2+2-2\sqrt{2}x&=3\\ \implies x^4-2x^2+1&=8x^2\\ \implies x^4-10x^2+1&=0 \end{aligned}$$

As $\sqrt{2}+\sqrt{3}$ is a root of the equation $x^4-10x^2+1=0$ with integer coefficients it is algebraic.

□

**Problem. 4.2.14** If $P(x)$ is a monic polynomial with integral coefficients and $k$ is any integer, must there exist an integer $m$ for which there are at at least $k$ distinct prime divisors of $P(m)$?

*Solution.* □

**Problem. 4.2.15**

(a) Factor $x^8+x^4+1$ into irreducible factors (i) over the rationals, (ii) over the reals, (iii) over the complex numbers.

(b) Factor $x^n-1$ over the complex numbers.

(c) Factor $x^4-2x^3+6x^2+22x+13$ over the complex numbers, given that $2+3i$ is a zero.

*Solution.* □

**Problem. 4.2.16**

(a) Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial of degree $n$ with integral coefficients. If $a_0$, $a_n$,and $f(1)$ are odd, prove that $f(x)=0$ has no rational roots.

(b) For what integer $a$ does $x^2-x+a$ divide $x^{13}+x+90$?

*Solution.* □

**Problem. 4.2.17**

(a) Suppose $f(x)$ is a polynomial over the real numbers and $g(x)$ is a divisor of $f(x)$ and $f'(x)$. Show that $(g(x))^2$ divides $f(x)$.

(b) Use the idea of part (a) to factor $x^6+x^4+3x^2+2x+2$ into a product of irreducibles over the complex numbers.

*Solution.* (a) We have $f(x) = g(x)q(x)$. From this we have, $f'(x) = g(x)q'(x) + g'(x)q(x)$. As $g(x)$ is also a divisor of $f'(x)$, we can easily see that $q(x)$ should be divisible by $g(x)$ as $g(x)$ cannot divide $g'(x)$ which is a polynomial of lower degree than $g(x)$. Therefore, $(g(x))^2$ divides $f(x)$.

(b) Let $f(x) = x^6 + x^4 + 3x^2 + 2x + 2$, we have $f'(x) = 6x^5 + 4x^3 + 6x + 2$.

□

**Problem. 4.2.18** Determine all pairs of positive integers $(m, n)$ such that $1 + x^n + x^{2n} + \cdots + x^{mn}$ is divisible by $1 + x + x^2 + \cdots + x^m$.

*Solution.* Denote the first and larger polynomial to be $f(x)$ and the second one to be $g(x)$. We could instead consider $f(x)$ modulo $g(x)$. Notice that $x^{m+1} = 1 \pmod{g(x)}$, and thus we can reduce the exponents of $f(x)$ to their equivalent modulo $m + 1$. We want the resulting $h(x)$ with degree less than $m + 1$ to be equal to $g(x)$ (of degree $m$), which implies that the exponents of $f(x)$ must be all different modulo $m + 1$. This can only occur if and only if $gcd(m + 1, n) = 1$. □

**Problem. 4.2.19**

(a) Let $F(x)$ be a polynomial over the real numbers. Prove that $a$ is a zero of multiplicity $m$

*Solution.* □

## §4.3 The Identity Theorem

**Problem. 4.3.11** Let $k$ be a positive integer. Find all polynomials

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

where the $a_i$ are real, which satisfy the equation

$$P(P(x)) = [P(x)]^k$$

**Problem. 4.3.12**

(a) Prove that $\log x$ cannot be expressed in the form $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials with real coefficients.

(b) Prove that $e^x$ cannot be expressed in the form $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials with real coefficients.

**Problem. 4.3.13** Show that

$$(1 + x)^n - x(1 + x)^n + x^2(1 + x)^n - \ldots x^k(1 + x)^n = (1 + x)^{n-1}(1 - (-x)^{k+1}),$$

and use this identity to prove that

$$\binom{n-1}{k} = \binom{n}{k} - \binom{n}{k-1} + \cdots \pm \binom{n}{0}$$



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*Solution.* We have

$$\begin{aligned}(1+x)^n - x(1+x)^n + x^2(1+x)^n - \dots x^k(1+x)^n &= (1+x)^n(1-x+x^2-\dots \pm x^k)\\ &= (1+x)^n\frac{(1-(-x)^{-k+1})}{1-(-x)}\\ &= (1+x)^{n-1}(1-(-x)^{k+1})\end{aligned}$$

The coefficient of $x^k$ on the left hand side is

$$\binom{n}{k} - \binom{n}{k-1} + \dots \pm \binom{n}{0}$$

as we only need the term $x^{k-i}$ from $(1+x)^n$ from the $i^{th}$ term. The coefficient of $x^k$ on the right hand side is

$$\binom{n-1}{k}$$

□

**Problem. 4.3.19**

(a) Solve the equation $x^3 - 3x^2 + 4 = 0$, given that two of its roots are equal.

(b) Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, given that its roots are in arithmetical progression.

*Solution.* (a) Let $f(x) = x^3 - 3x^2 + 4$. As $f(x) = 0$ has two equal roots, $f(x) = 0$ and $f'(x) = 0$ share that root. The roots of $f'(x) = 3x^2 - 6x = 0$ are $x = 0$ and $x = 2$. As $x = 0$ is not a root of $f(x) = 0$, $x = 2$ is a common root of $f(x)$ and $f'(x)$.As the product of the roots of $f(x) = 0$ is $-4$, the roots of $f(x) = 0$ are $-1, 2$ and 2.

(b) Let $a - d, a, a + d$ be the roots of $x^3 - 9x^2 + 23x - 15 = 0$. We have sum of the roots is $3a = 9$ and the product of the roots is $3(9 - d^2) = 15$. Therefore, the roots of the equation are $1, 3$ and 5.

□

**Problem. 4.3.20** Given $r, s, t$ are the roots of $x^3 + ax^2 + bx + c = 0$,

(a) Evaluate $1/r^2 + 1/s^2 + 1/t^2$, provided that $c \neq 0$.

(b) Find a polynomial whose roots are $r^2, s^2, t^2$

*Solution.* (a) Let $f(x) = x^3 + ax^2 + bx + c$.As $c \neq 0$, none of the roots of $f(x) = 0$ can be zero.We now need to find $g$ such that $y = 1/z^2$ is a root of $g(y) = 0$ where $z$ is the root of $f(x) = 0$.Therefore, $1/r^2 + 1/s^2 + 1/t^2$ will be the sum of the roots of $g(y) = 0$. We have $f(1/\sqrt{y}) = 0$, which means

$$\begin{aligned}&(\frac{1}{\sqrt{y}})^3 + a(\frac{1}{\sqrt{y}})^2 + b(\frac{1}{\sqrt{y}}) + c = 0\\ \implies &c^2y^3 + y^2(2ac - b^2) + y(a^2 - 2b) - 1 = 0\end{aligned}$$

Therefore, $1/r^2 + 1/s^2 + 1/t^2 = b^2 - 2ac$.

(b) We now need to find $h(y)$ such that $y = z^2$ is a root of $h(y) = 0$ where $z$ is the root of $f(x) = 0$. We have $f(\sqrt{y}) = 0$, which means

$$\begin{aligned}&\sqrt{y}^3 + a\sqrt{y}^2 + b\sqrt{y} + c = 0\\ \implies &y^3 + y^2(2b - a^2) + y(b^2 - 2ac) - c^2 = 0\end{aligned}$$

□

# §4.4 Abstract Algebra

**Problem. 4.4.12** Let $G$ be a set, and $*$ a binary operation on $G$ which is associative and is such that for all $a, b$ in $G$, $a^2b = b = ba^2$(suppressing the $*$). Show that $G$ is a commutative group.

**Problem. 4.4.13** $A$ is a subset of a finite group $G$, and $A$ contains more than one-half of the elements of $G$. Prove that each element of $G$ is the product of two elements of $A$.

# 5 Summation Of Series

**Problem. 5.1.13** Prove that

$$\sum_{k=0}^{n}\left[\frac{n-2k}{n}\binom{n}{k}^2\right]=\frac{2}{n}\binom{2n-2}{n-1}$$

*Solution.* We have

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n} \tag{5.1}$$

$$\binom{r}{0}\binom{s}{n}+\binom{r}{1}\binom{s}{n+1}+\binom{r}{2}\binom{s}{n+2}+\cdots+\binom{r}{n}\binom{s}{n+n}=\binom{r+s}{s-n} \tag{5.2}$$

Using 5.1 and setting $r=n-1$, $s=n$ and $n=1$ in 5.2, we have

$$\begin{aligned}\sum_{k=0}^{n}\left[\frac{n-2k}{n}\binom{n}{k}\right]^2&=\sum_{k=0}^{n}\left(\binom{n}{k}^2+\frac{4k^2}{n^2}\binom{n}{k}^2-\frac{4k}{n}\binom{n}{k}^2\right)\\&=\binom{2n}{n}+4\sum_{k=1}^{n}\binom{n-1}{k-1}^2+4-4\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{n}{k}-4\\&=\binom{2n}{n}+4\binom{2n-2}{n-1}-4\binom{2n-1}{n-1}\\&=\frac{(2n-2)!}{n!(n+1)!}\left(2n(2n-1)(n+1)+4n^2(n+1)-4(2n-1)n(n+1)\right)\\&=\frac{(2n-2)!}{n!(n+1)!}2n(n+1)=\frac{2}{n}\binom{2n-2}{n-1}\end{aligned}$$

□

**Problem. 5.1.14**

**Problem. 5.2.8** Sum the series $1+22+333+\cdots+n(\underbrace{11\ldots1}_{n})$.

*Solution.* We have

$$\begin{aligned}S&=1+22+333+\cdots+n(\underbrace{11\ldots1}_{n})\\&=\frac{1}{9}\{9+2\cdot99+\cdots+n(\underbrace{99\ldots9}_{n})\}\\&=\frac{1}{9}\{10-1+2(10^2-1)+\cdots+n(10^n-1)\}\\&=\frac{1}{9}\left(\frac{n10^{n+2}-(n+1)10^{n+1}+10}{81}-\frac{n(n+1)}{2}\right)\end{aligned}$$

□

**Problem. 5.4.12** Let $p$ and $q$ be real numbers with $1/p-1/q=1, 0<p\leqslant\frac{1}{2}$. Show that

$$p+\frac{1}{2}p^2+\frac{1}{3}p^3+\cdots=q-\frac{1}{2}q^2-\frac{1}{3}q^3+\cdots.$$

*Solution.* We have

$$\frac{1}{p} - \frac{1}{q} = 1 \implies p = \frac{q}{1+q}$$

Differentiating the above, we have

$$-\frac{dp}{p^2} + \frac{dq}{q^2} = 0 \implies dp = \frac{1}{(1+q)^2} dq$$

We also have

$$\int \frac{1}{1-p} dp = \int \frac{1}{1 - \frac{q}{1+q}} \frac{1}{(1+q)^2} dq = \int \frac{1}{1+q} dq$$

Integrating the series expansions on both sides, we have

$$\begin{aligned} &\int 1 + p + p^2 + \cdots dp = \int 1 - q + q^2 - q^3 + \cdots dq \\ \implies\ & p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \cdots = q - \frac{1}{2}q^2 - \frac{1}{3}q^3 + \cdots \end{aligned}$$

□

# 6 Real Analysis

## §6.1 Continuous Functions

## §6.2 Intermediate-Value Theorem

**Problem. 6.2.4** Suppose that $f : [0,1] \to [0,1]$ is continuous.Prove that there exists a number $c$ in $[0,1]$ such that $f(c) = c$.

*Solution.* If $f(0) = 0$ or $f(1) = 1$, there is nothing to prove. Consider the function $g(x) = f(x) - x$. It is easy to see that $g(x)$ is continuous on $[0,1]$. We also have $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ as $f(0) \neq 0$ and $f(1) \neq 1$. Intermediate-Value theorem guarantees the existence of a $c$ in $[0,1]$ for which $g(c) = 0$. Therefore, there exists a number $c$ in $[0,1]$ such that $f(c) = c$. □

**Problem. 6.2.4** A rock climber starts to climb a mountain at $7:00$A.M. on Saturday and gets to the top at $5:00$P.M. He camps on top and climbs back down on Sunday, starting at $7:00$A.M. and getting back to his original starting point at $5:00$P.M. Show that at some time of day on Sunday he was at the same elevation as he was at that time on Saturday.

*Solution.* We have two functions $h_a(t)$ and $h_d(t)$ giving the height of the climber as a function of time for the ascent and descent. Let $t_s$ and $t_e$ be $7:00$A.M. and $5:00$P.M. respectively. We can assume that $h_a$ and $h_d$ are continuous on $[t_s, t_e]$. Define a function $h(t) = h_a(t) - h_d(t)$. We have $h(t_s) < 0$ and $h(t_e) < 0$. Intermediate-Value theorem guarantees the existence of a $t$ in $[t_s, t_e]$ for which $h(t) = 0$. □

**Problem. 6.2.6** Prove that a continuous function which takes on no value more than twice must take on some value exactly once.

*Solution.* □

## §6.3 Rolle's Theorem

**Problem. 6.5.5**

(a) Show that $5x^4 - 4x + 1 = 0$ has a root between 0 and 1.

(b) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

show that the equation $a_0 + a_1 x + \cdots + a_n x^n = 0$ has atleast one real root.

*Solution.* We use Rolle's theorem for the following:

(a) Let $f(x) = x^5 - 2x^2 + x$. f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. We also have $f(0) = 0$ and $f(1) = 0$.Therefore there is a number c in $(0, 1)$ such that $f'(c) = 5c^4 - 4x + 1 = 0$.

(b) Let $f(x)$ be

$$a_0 x + \frac{a_1 x^2}{2} + \cdots + \frac{a_n x^{n+1}}{n+1} = 0$$

f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. We also have $f(0) = 0$ and $f(1) = 0$.Therefore there is a number c in $(0, 1)$ such that $f'(c) = a_0 + a_1 c + \cdots + a_n c^n = 0$.

□

**Problem. 6.5.6**

(a) Suppose that $f : [0, 1] \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f(x) > 0$ for $x$ in $(0, 1)$. Prove that there is a number c in $(0, 1)$ such that

$$\frac{2f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}$$

(b) Is there a number d in $(0, 1)$ such that

$$\frac{3f'(d)}{f(d)} = \frac{f'(1-d)}{f(1-d)}$$

*Solution.* We use Rolle's theorem for the following:

(a) Consider $g(x) = f^2(x)f(1 - x)$. g is differentiable on $(0, 1)$.

We also have $g(0) = 0$ and $g(1) = 0$ .

Therefore there is a number c in $(0, 1)$ such that $g'(c) = -f^2(c)f'(1 - c) + 2f(c)f'(c)f(1 - c) = 0$. Therefore, as $f(x) > 0$ for $x > 0$,

$$\frac{2f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}$$

(b) Consider $g(x) = f^3(x)f(1 - x)$. g is differentiable on $(0, 1)$.

We also have $g(0) = 0$ and $g(1) = 0$ .

Therefore there is a number $d$ in $(0,1)$ such that $g'(d) = -f^3(d)f'(1-d) + 3f^2(d)f'(d)f(1-d) = 0$. Therefore, as $f(x) > 0$ for $x > 0$,

$$\frac{3f'(d)}{f(d)} = \frac{f'(1-d)}{f(1-d)}$$

□

**Problem. 6.5.7**

(a) *Cauchy mean-value theorem* If $f$ and $g$ are continuous on $[a,b]$ and differentiable on $(a,b)$, then there is a number $c$ in $(a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

(b) *Mean-value theorem* If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there is a number $c$ in $(a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

*Solution.* (a) Define $F(x)$ as follows

$$F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

We see that $F(a) = F(b)$. Rolle's theorem guarantees the existence of $c$ in $(a,b)$ such that $F'(c) = 0$. Therefore,

$$\begin{aligned} f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0 \\ \implies [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c) \end{aligned}$$

(b) Setting $g(x) = x$ in Cauchy's mean value theorem we have

$$\begin{aligned} [f(b) - f(a)] = [b - a]f'(c) \\ \implies \frac{f(b) - f(a)}{b - a} = f'(c) \end{aligned}$$

□

**Problem. 6.5.8**

(a) Show that $x^3 - 3x + b = 0$ cannot have more than one zero in $[-1,1]$, regardless of the value of $b$.

(b) Let $f(x) = (x^2 - 1)e^{cx}$. Show that $f'(x) = 0$ for exactly one $x$ in the interval $(-1,1)$ and that this $x$ has the same sign as parameter $c$.

*Solution.* A useful corollary to Rolle's theorem is that if $f$ is a continuous and differentiable function, say on the interval $[a,b]$, and if $x_1$ and $x_2$ are zeros of $f$, $a < x_1 < x_2 < b$, then $f'$ has a zero between $x_1$ and $x_2$. More generally, if $f$ has $n$ distinct zeros in $[a,b]$, then $f'$ has at least $n-1$ zeros(these are interlaced with the zeros of $f$), $f''$ has at least $n-2$ zeros(assuming $f'$ is continuous and differentiable on $[a,b]$), and so forth.

(a) If $f(x) = x^3 - 3x + b$ has more than one zero in $[-1, 1]$, then $f'(x)$ should have at least one zero in $(-1, 1)$. But $f'(x) = 3x^2 - 3$ has no zeros in $(-1, 1)$, therefore $f(x)$ cannot have more than zero in $[-1, 1]$ irrespective of the value of $b$.

(b) We have $f(-1) = f(1) = 0$. Therefor, $f'(x)$ has at least one zero in $(-1, 1)$. As the zeros of $f'(x)$ have to be interlaced with $f(x)$, $f'(x)$ cannot have more than one zero in $(-1, 1)$. If $d$ is a root of $f'(x) = 0$. we have

$$(d^2 - 1)ce^{cd} + e^{cd}2d = 0$$
$$\implies c = \frac{2d}{1 - d^2}$$

As $(1 - d^2)$ is positive for $d$ in $(-1, 1)$, $d$ the zero of $f'(x)$ has the same sign as the parameter $c$.

□

**Problem. 6.5.9** How many zeros does the function $f(x) = 2^x - 1 - x^2$ have on the real line?

*Solution.* Clearly there are no roots for negative $x$, since for such $x$, $2^x < 1$, whereas $1 + x^2 > 1$. There are certainly roots at $x = 0$ and 1. Also $2^4 < 4^2 + 1$, whereas $2^5 > 5^2 + 1$, so there is a root between 4 and 5. We have to show that there are no other roots. Put $f(x) = 2^x - x^2 - 1$. Then $f''(x) = (\ln 2)^2 2^x - 2$. This is strictly increasing with a single zero. $f'(0) > 0$, so $f'(x)$ starts positive, decreases through zero to a minimum, then increases through zero. So it has just two zeros. Hence $f(x)$ has at most three zeros, which we have already found. □

**Problem. 6.5.10** Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients such that $f$ has $n + 1$ distinct real zeros. Use Rolle's therorem to show that $a_k = 0$, for $0 \leqslant k \leqslant n$.

*Solution.* If $f$ has $n + 1$ distinct real zeros, the equation $f^n(x) = n!a_n = 0$ where $f^n(x)$ is the $n^{th}$ derivative should have at least one zero. This gives us $a_n = 0$.By extending the same argument to the other derivatives $f^k(x)$ where $1 \leqslant k \leqslant n - 1$, we can show that all the coefficients of $f(x)$ are zero. □

**Problem. 6.5.11** If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, prove that there is a root of $f'(x) - af(x) = 0$ between any two roots of $f(x) = 0$.

*Solution.* □

## §6.4 Mean-Value Theorem


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## §6.5 L'Hôspital's Rule

**Problem. 6.7.3** Evaluate

$$\lim_{n\to\infty} 4^n \left(1 - \cos\frac{\theta}{2^n}\right)$$

*Solution.* We have

$$\lim_{n\to\infty} 4^n \left(1 - \cos\frac{\theta}{2^n}\right) = \lim_{n\to\infty} \frac{2\sin^2\frac{\theta}{2^{n+1}}}{4\frac{\theta^2}{4^{n+1}}}\theta^2 = \frac{\theta^2}{2}$$

□

**Problem. 6.7.4** Evaluate the following limits

(a) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

(b) $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$

(c) $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$

(d) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

(e) $\lim_{n\to\infty} \frac{2p_n P_n}{p_n + P_n}$, where $p_n = \left(1 + \frac{1}{n}\right)^n$ and $P_n = \left(1 + \frac{1}{n}\right)^{n+1}$

*Solution.* We use L'Hôspital's Rule for the following

(a) Let $y = \left(1 + \frac{1}{n}\right)^n$. We have

$$\begin{aligned}
\log(y) &= n\log\left(1 + \frac{1}{n}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} n\log\left(1 + \frac{1}{n}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}} \\
\implies \log \lim_{n\to\infty} y &= 1 \\
\implies \lim_{n\to\infty} y &= e
\end{aligned}$$

(b) Let $y = \left(\frac{n+1}{n+2}\right)^n$. We have

$$\begin{aligned}
\log(y) &= n\log\left(\frac{n+1}{n+2}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} n\log\left(\frac{n+1}{n+2}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{n+2}}\frac{1}{(n+2)^2}}{\frac{-1}{n^2}} \\
\implies \log \lim_{n\to\infty} y &= -1 \\
\implies \lim_{n\to\infty} y &= e^{-1}
\end{aligned}$$

(c) Let $y = \left(1 + \frac{1}{n^2}\right)^n$. We have

$$\begin{aligned}
\log(y) &= n\log\left(1 + \frac{1}{n^2}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} n\log\left(1 + \frac{1}{n^2}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(\frac{-2}{n^3}\right)}{\frac{-1}{n^2}} \\
\implies \log \lim_{n\to\infty} y &= 0 \\
\implies \lim_{n\to\infty} y &= 1
\end{aligned}$$

(d) Let $y = \left(1 + \frac{1}{n}\right)^{n^2}$. We have

$$\begin{aligned}
\log(y) &= n^2\log\left(1 + \frac{1}{n}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} n^2\log\left(1 + \frac{1}{n}\right) \\
\implies \log \lim_{n\to\infty} y &= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\left(\frac{-1}{n^2}\right)}{\frac{-2}{n^3}} \\
\implies \log \lim_{n\to\infty} y &= \infty \\
\implies \lim_{n\to\infty} y &= \infty
\end{aligned}$$

(e) We have

$$\begin{aligned}
\lim_{n\to\infty} p_n &= e \\
\lim_{n\to\infty} P_n &= e
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{2p_nP_n}{p_n + P_n} = \frac{2e^2}{2e} = e$$

□

**Problem. 6.7.5** Let $0 < a < b$. Evaluate

$$\lim_{t\to 0}\left[\int_0^1 [bx + a(1-x)]^t dx\right]^{1/t}$$

*Solution.* We first evaluate the integral.Let $y = bx + a(1-x)$, we have $dy = (b-a)dx$.

$$\int_0^1 [bx + a(1-x)]^t dx = \frac{1}{b-a}\int_a^b y^t dy = \frac{1}{b-a}\left(\frac{b^{t+1}}{t+1} - \frac{a^{t+1}}{t+1}\right)$$

$$
\begin{aligned}
\text{Let } z &= \left[\frac{1}{b-a}\left(\frac{b^{t+1}}{t+1}-\frac{a^{t+1}}{t+1}\right)\right]^{1/t} \\
\implies \log(z) &= \frac{1}{t}\log\left[\frac{1}{b-a}\left(\frac{b^{t+1}}{t+1}-\frac{a^{t+1}}{t+1}\right)\right] \\
\implies \log\lim_{t\to 0} z &= \lim_{t\to 0}\frac{1}{\frac{1}{b-a}\left(\frac{b^{t+1}}{t+1}-\frac{a^{t+1}}{t+1}\right)}\frac{1}{b-a}\left(\frac{b^{t+1}((t+1)\log(b)-1)}{(t+1)^2}-\frac{a^{t+1}((t+1)\log(a)-1)}{(t+1)^2}\right) \\
\implies \log\lim_{t\to 0} z &= \frac{b\cdot\log(b)-a\cdot\log(a)}{b-a}-1 \\
\implies \lim_{t\to 0} z &= \frac{b^{\frac{b}{b-a}}}{ea^{\frac{a}{b-a}}}
\end{aligned}
$$

□

**Problem. 6.7.6** Calculate

$$\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}dt$$

*Solution.* We have

$$
\begin{aligned}
\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}dt = \lim_{x\to\infty}\frac{x\int_0^x e^{t^2}dt}{e^{x^2}} &= \lim_{x\to\infty}\frac{xe^{x^2}+\int_0^x e^{t^2}dt}{e^{x^2}2x} \\
&= \frac{1}{2}+\lim_{x\to\infty}\frac{\int_0^x e^{t^2}dt}{e^{x^2}2x} \\
&= \frac{1}{2}+\lim_{x\to\infty}\frac{e^{x^2}}{2e^{x^2}+4x^2e^{x^2}} = \frac{1}{2}
\end{aligned}
$$

□

**Problem. 6.7.7** Prove that the function $y=(x^2)^x$, $y(0)=1$, is continuous at $x=0$.

*Solution.* We need to prove that $\lim_{x\to 0}(x^2)^x = 1$.

$$
\begin{aligned}
\text{Let } y &= (x^2)^x \\
\implies \log(y) &= 2x\log(x) \\
\implies \log\lim_{x\to 0} y &= 2\lim_{x\to 0}\frac{\log(x)}{\frac{1}{x}} \\
\implies \log\lim_{x\to 0} y &= 2\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0 \\
\implies \lim_{x\to 0} y &= 1
\end{aligned}
$$

□

## §6.6 The Fundamental Theorem

**Problem. 6.9.7** What function is defined by the equation

$$f(x) = \int_0^x f(t)dt + 1$$

*Solution.* Differentiating both sides we get

$$f'(x) = f(x)$$

The solution for the above differential equation is

$$f(x) = ke^x$$

We have $f(0) = 1$, therefore $k = 1$. □

**Problem. 6.9.8** Let $f : [0, 1] \to (0, 1)$ be continuous.Show that the equation

$$2x - \int_0^x f(t)dt = 1$$

has one and only one solution in the interval $[0, 1]$.

*Solution.* Let $F(x) = 2x - \int_0^x f(t)dt - 1$. $F(x)$ is continuous on $[0, 1]$. We have $F(0) = -1$ and $F(1) = 1 - \int_0^1 f(t)dt > 0$. Intermediate-Value Theorem guarantees the existence of value $c$ in $[0, 1]$ such that $F(c) = 0$. We have $F'(x) = 2 - f(x) > 0$ for $x$ in $[0, 1]$ which means $F(x)$ is strictly increasing on $[0, 1]$. Therefore, $F(x)$ cannot intersect the $x - axis$ more than once. □

**Problem. 6.9.9** Suppose that $f$ is a continuous function for all $x$ which satisfies the equation

$$\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$$

where $C$ is a constant.Find an explicit form of $f(x)$ and find the value of the constant $C$.

*Solution.* Differentiating wrt $x$ on both sides, we have

$$\begin{aligned} & f(x) = -x^2 f(x) + 2x^{15}(1 + x^2) \\ \implies & f(x) = 2x^{15} \end{aligned}$$

We also have $C + \int_0^1 x^2 2x^{15} dx = 0$. Therefore $C = -\frac{1}{9}$. □

**Problem. 6.9.10** Let $C_1$ and $C_2$ be curves passing through the origin.A curve $C$ is said to *bisect in area* the region between $C_1$ and $C_2$ if for each point $P$ of $C$ the two shaded areas $A$ and $B$ shown in the figure have equal areas. Determine the upper curve $C_2$ given that the bisecting curve has the equation $y = x^2$ and the lower curve $C_1$ has the equation $y = \frac{x^2}{2}$.

*Solution.* Let $x = f(y)$ be the equation of the upper curve. We have

$$\int_0^u t^2 - \frac{t^2}{2} dt = \int_0^{u^2} \sqrt{t} - f(t)dt$$

Differentiating both sides wrt $u$ we get,

$$
\begin{aligned}
\frac{u^2}{2} &= \left(u - f(u^2)\right) 2u \\
\implies f(u^2) &= \frac{3}{4}u \\
\implies f(y) &= \frac{3}{4}\sqrt{y} \\
\implies x &= \frac{3}{4}\sqrt{y} \\
\implies x^2 &= \frac{9}{16}y \\
\implies y &= \frac{16}{9}x^2 \text{ is the upper curve } C_2
\end{aligned}
$$

□

**Problem. 6.9.11** Sum the series $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \dots$

*Solution.* Consider the function defined by the infinite series

$$f(x) = x + \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^{13}}{13} - \dots$$

for $0 < x \leqslant 1$. The series is absolutely convergent for $|x| < 1$,

and therefore we can rearrange the terms:

$$f(x) = \left(x + \frac{x^9}{9} + \dots\right) + \left(\frac{x^3}{3} + \frac{x^{11}}{11} + \dots\right) - \left(\frac{x^5}{5} + \frac{x^{13}}{13} + \dots\right) - \left(\frac{x^7}{7} + \frac{x^{15}}{15} + \dots\right) + \dots$$

We have for $0 < x < 1$,

$$
\begin{aligned}
f'(x) &= (1 + x^8 + \dots) + (x^2 + x^{10} + \dots) - (x^4 + x^{12} + \dots) - (x^6 + x^{14} + \dots) + \dots \\
&= (1 + x^2 - x^4 - x^6)(1 + x^8 + \dots) \\
&= \frac{(1 + x^2)(1 - x^4)}{1 - x^8} = \frac{1 + x^2}{1 + x^4}
\end{aligned}
$$

Integrating and noting that $f(0) = 0$, we get

$$f(x) = \frac{\arctan(\sqrt{2}x + 1) - \arctan(1 - \sqrt{2}x)}{\sqrt{2}}$$

Therefore,

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \dots = f(1) = \frac{\pi}{2\sqrt{2}}$$

□

**Problem. 6.9.12** Suppose that $f$ is differentiable, and that $f'(x)$ is strictly increasing for $x \geqslant 0$. If $f(0) = 0$, prove that $f(x)/x$ is strictly increasing for $x > 0$.

*Solution.* □

# 7 Inequalities

## §7.1 Arithmetic-Mean-Geometric-Mean Inequality

**Problem. 7.2.6** If $a, b, c$ are positive numbers

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geqslant 9a^2b^2c^2$$

*Solution.* We have from $AM \geqslant GM$,

$$\frac{a^2b + b^2c + c^2a}{3} \geqslant abc$$
$$\frac{ab^2 + bc^2 + ca^2}{3} \geqslant abc$$

Multiplying the above, we get

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geqslant 9a^2b^2c^2$$

□

**Problem. 7.2.7** Suppose $a_1, \ldots, a_n$ are positive numbers and $b_1, \ldots, b_n$ is a rearrangement of $a_1, \ldots, a_n$, Show that

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geqslant n$$

*Solution.* We have from $AM \geqslant GM$,

$$\frac{\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}}{n} \geqslant \sqrt[n]{\frac{a_1}{b_1} \cdots \frac{a_n}{b_n}}$$

As $b_1, \ldots, b_n$ is a rearrangement of $a_1, \ldots, a_n$, the product on the right hand side above is 1. Therefore,

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \geqslant n$$

□

**Problem. 7.2.9** For each integer $n > 2$, prove that

(a) $\prod_{k=0}^{n} \binom{n}{k} < \left(\frac{2^n-2}{n-1}\right)^{n-1}$

(b) $n! < \left(\frac{n+1}{2}\right)^n$

(c) $1x3x5x \cdots x(2n-1) < n^n$

*Solution.* Using $AM > GM$

(a) We have

$$\frac{\sum_{k=1}^{n-1}\binom{n}{k}}{n-1} > \left(\prod_{k=0}^{n}\binom{n}{k}\right)^{\frac{1}{n-1}}$$
$$\implies \left(\frac{2^n-2}{n-1}\right)^{n-1} > \prod_{k=0}^{n}\binom{n}{k}$$

(b) We have

$$\frac{\sum_{k=1}^{n} k}{n} > (n!)^{\frac{1}{n}}$$
$$\implies \left(\frac{n+1}{2}\right)^{n} > n!$$

(c) We have

$$\frac{\sum_{k=1}^{n} 2k-1}{n} > (1x3x5x\cdots x(2n-1))^{\frac{1}{n}}$$
$$\implies (\frac{n^2}{n})^n = n^n > 1x3x5x\cdots x(2n-1)$$

□

**Problem. 7.2.10** Given that all roots of $x^6-6x^5+ax^4+bx^3+cx^2+dx+1=0$ are positive, find $a, b, c, d$.

*Solution.* Let $u, v, w, x, y, z$ be the roots of the above equation. From $AM \geqslant GM$, we have

$$\frac{u+v+w+x+y+z}{6} = \frac{6}{6} \geqslant (uvwxyz)^{1/6} = 1$$

As we have equality only when $u = v = w = x = y = z = 1$, we have

$$x^6-6x^5+ax^4+bx^3+cx^2+dx+1 \equiv (x-1)^6 = x^6-6x^5+15x^4-20x^3+15x^2-6x+1$$

Therefore $a = 15, b = -20, c = 15, d = -6$. □

## §7.2 Cauchy-Schwarz Inequality

**Problem. 7.3.6** Use the Cauchy-Schwarz inequality to prove that if $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1+a_2+\cdots+a_n = 1$, then $a_1^2+a_2^2+\cdots+a_n^2 \geqslant 1/n$.

*Solution.* We have

$$(a_1\cdot 1+\cdots+a_n\cdot 1)^2 \leqslant (a_1^2+\cdots+a_n^2)(1^2+\cdots+1^2)$$
$$\implies 1 \leqslant (a_1^2+\cdots+a_n^2)n$$
$$\implies \frac{1}{n} \leqslant (a_1^2+\cdots+a_n^2)$$

□

**Problem. 7.3.7** Use the Cauchy-Schwarz inequality to prove the following

(a) if $p_1, p_2, \dots, p_n, x_1, x_2, \dots, x_n$ are $2n$ positive real numbers,

$$(p_1x_1 + \cdots + p_nx_n)^2 \leqslant (p_1 + \cdots + p_n)(p_1x_1^2 + \cdots + p_nx_n^2)$$

(b) If $a, b, c$ are positive numbers

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geqslant 9a^2b^2c^2$$

(c) If $x_k, y_k, k = 1, 2, \dots, n$ are positive numbers,

$$\sum_{k=1}^{n} x_k y_k \leqslant \left(\sum_{k=1}^{n} k x_k^2\right)^{1/2} \left(\sum_{k=1}^{n} y_k^2/k\right)^{1/2}$$

(d) If $a_k, b_k, c_k, k = 1, 2, \dots, n$ are positive numbers,

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leqslant \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right)\left(\sum_{k=1}^{n} c_k^2\right)^2$$

(e) If $C_k = \binom{n}{k}$ for $n > 2, 1 \leqslant k \leqslant n$,

$$\sum_{k=1}^{n} \sqrt{C_k} \leqslant \sqrt{n(2^n - 1)}$$

*Solution.* Using Cauchy Schwarz inequality

(a) We have

$$\begin{aligned}
(\sqrt{p_1}\sqrt{p_1}x_1 + \cdots + \sqrt{p_n}\sqrt{p_n}x_n)^2 &\leqslant \left(\sqrt{p_1}^2 + \cdots + \sqrt{p_n}^2\right)\left((\sqrt{p_1}x_1)^2 + \cdots + (\sqrt{p_n}x_n)^2\right) \\
\implies (p_1x_1 + \cdots + p_nx_n)^2 &\leqslant (p_1 + \cdots + p_n)(p_1x_1^2 + \cdots + p_nx_n^2)
\end{aligned}$$

(b) We have

$$\begin{aligned}
\left((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2\right)\left((c\sqrt{b})^2 + (a\sqrt{c})^2 + (b\sqrt{a})^2\right) &\geqslant \left(a\sqrt{b}\cdot c\sqrt{b} + b\sqrt{c}\cdot a\sqrt{c} + c\sqrt{a}\cdot b\sqrt{a}\right)^2 \\
\implies (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) &\geqslant (3abc)^2 = 9a^2b^2c^2
\end{aligned}$$

(c) We have

$$\begin{aligned}
\left(\sum_{k=1}^{n} (\sqrt{k}x_k)^2\right)\left(\sum_{k=1}^{n} (y_k/\sqrt{k})^2\right) &\geqslant \left(\sum_{k=1}^{n} \sqrt{k}x_k \cdot y_k/\sqrt{k}\right)^2 \\
\implies \left(\sum_{k=1}^{n} k x_k^2\right)^{1/2}\left(\sum_{k=1}^{n} y_k^2/k\right)^{1/2} &\geqslant \sum_{k=1}^{n} x_k y_k
\end{aligned}$$

(d) We have

$$\begin{aligned}
\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 &\leqslant \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right) \\
\implies \left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 &\leqslant \left(\sum_{k=1}^{n} a_k^4\right)^{1/2}\left(\sum_{k=1}^{n} b_k^4\right)^{1/2}\left(\sum_{k=1}^{n} c_k^2\right) \\
\implies \left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 &\leqslant \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^4\right)\left(\sum_{k=1}^{n} c_k^2\right)^2
\end{aligned}$$

(e) We have

$$\begin{aligned}
\left(\sum_{k=1}^{n} 1 \cdot \sqrt{C_k}\right)^2 &\leqslant \left(\sum_{k=1}^{n} 1^2\right)\left(\sum_{k=1}^{n} C_k\right) \\
\implies \sum_{k=1}^{n} \sqrt{C_k} &\leqslant \sqrt{n(2^n-1)}
\end{aligned}$$

□

**Problem. 7.3.8** For $n$ a positive integer, let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two (not necessarily distinct) permutations of $(1, 2, \ldots, n)$. Find sharp lower and upper bounds for $a_1b_1 + \cdots + a_nb_n$.

*Solution.* We have

$$a_1b_1 + \cdots + a_nb_n \leqslant \left(\sum_{k=1}^{n} k^2\right)^{1/2}\left(\sum_{k=1}^{n} k^2\right)^{1/2} = \frac{n(n+1)(2n+1)}{6}$$

Using the rearrangement inequality we have

$$\begin{aligned}
a_1b_1 + \cdots + a_nb_n \geqslant 1 \cdot n + \cdots + 1 \cdot n &= \sum_{k=1}^{n} k(n+1-k) \\
&= \frac{n(n+1)}{2}\left(n+1-\frac{2n+1}{3}\right) \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

□

**Problem. 7.3.9** If $a, b, c, d$ are positive numbers such that $c^2 + d^2 = (a^2 + b^2)^3$ prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geqslant 1$$

with equality if and only if $ad = bc$.

*Solution.* We have

$$\begin{aligned}
(a^2+b^2)^3 &= c^2 + d^2 \\
\implies (a^2+b^2)^4 = (a^2+b^2)(c^2+d^2) &\geqslant (ac+bd)^2 \\
\implies (a^2+b^2)^2 &\geqslant (ac+bd)
\end{aligned}$$

We also have

$$\begin{aligned}\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd) &\geqslant (a^2+b^2)^2 \geqslant (ac+bd)\\ \implies \left(\frac{a^3}{c}+\frac{b^3}{d}\right) &\geqslant 1\end{aligned}$$

□

**Problem. 7.3.10** Let $P$ be a point in the interior of triangle $ABC$, and let $r_1, r_2, r_3$ denote the distances from $P$ to the sides $a_1, a_2, a_3$ of the triangle respectively. Use Cauchy-Schwarz inequality to show that the minimum value of

$$\frac{a_1}{r_1}+\frac{a_2}{r_2}+\frac{a_3}{r_3}$$

occur when $P$ is at the incenter of triangle $ABC$.

*Solution.* We have

$$\begin{aligned}\left(\left(\sqrt{\frac{a_1}{r_1}}\right)^2+\left(\sqrt{\frac{a_2}{r_2}}\right)^2+\left(\sqrt{\frac{a_3}{r_3}}\right)^2\right)\left((\sqrt{a_1r_1})^2+(\sqrt{a_2r_2})^2+(\sqrt{a_3r_3})^2\right) &\geqslant (a_1+a_2+a_3)^2\\ \implies \frac{a_1}{r_1}+\frac{a_2}{r_2}+\frac{a_3}{r_3} &\geqslant \frac{4s^2}{2\Delta}\\ \implies \frac{a_1}{r_1}+\frac{a_2}{r_2}+\frac{a_3}{r_3} &\geqslant \frac{2s}{r}=\frac{a_1}{r}+\frac{a_2}{r}+\frac{a_3}{r}\end{aligned}$$

where $r$ is the inradius, $\Delta$ is the area of $ABC$ and $s$ is the semi-perimeter. □

# 8 Geometry

## §8.1 Classical Plane Geometry

**Problem. 8.1.8** If $a, b, c$ are the sides of a triangle $ABC$, $t_a, t_b, t_c$ are the angle bisectors, and $T_a, T_b, T_c$ are the angle bisectors extended until they are chords of the circle circumscribing the triangle $ABC$, prove that

$$abc = \sqrt{T_a T_b T_c t_a t_b t_c}$$

*Solution.* □

**Problem. 8.1.12** We are given an inscribed triangle $ABC$. Let $R$ denote the circumradius; let $h_a$ denote the altitude AD.

(a) Show that triangles $ABD$ and $ALC$ are similar, and hence that $h_a = 2R = bc$.

(b) Show that the area of $\Delta ABC$ is $abc/4R$.

*Solution.* (a) In triangles $ABC$ and $ALC$, $\angle ABC$ is equal to $\angle ALC$ and $\angle ACL = \angle BAC = 90°$. Therefore triangles $ABD$ and $ALC$ are similar.We also have

$$\frac{c}{h_a} = \frac{2R}{b}$$
$$\implies h_a = \frac{bc}{2R}$$

(b) Area of the triangle $ABC$ is

$$\frac{1}{2}ah_a = \frac{1}{2}a\frac{bc}{2R} = \frac{abc}{4R}$$

□

**Problem. 8.1.13** The radius of the inscribed circle of a triangle is 4,and the segments into which one side is divided by the point of contact are 6 and 8.Determine the other two sides.

*Solution.* The sides of the triangle $14, 6 + x$ and $8 + x$.Hence, $s = (14 + 6 + x + 8 + x)/2 = 14 + x$.

$$\Delta = \sqrt{(14 + x) \cdot x \cdot 8 \cdot 6} = 4(14 + x)$$
$$\implies 14 + x = 3x$$
$$\implies x = 7$$

The other two sides of the triangle are 13 and 21. □

**Problem. 8.1.14** Triangles $ABC$ and $DEF$ are inscribed in the same circle.Prove that

$$\sin A + \sin B + \sin C = \sin D + \sin E + \sin F$$

if and only if the perimeters of the given triangles are equal.

Let $a_i, b_i, c_i$ for $i = 1, 2$ be the sides of the two triangles and $R$ the circumradius. We have,

*Solution.*

$$\begin{aligned} \sin A + \sin B + \sin C &= \sin D + \sin E + \sin F \\ \iff \frac{a_1}{2R} + \frac{b_1}{2R} + \frac{c_1}{2R} &= \frac{a_2}{2R} + \frac{b_2}{2R} + \frac{c_2}{2R} \\ \iff a_1 + b_1 + c_1 &= a_2 + b_2 + c_2 \end{aligned}$$

□

**Problem. 8.1.15** In the following figure, $CD$ is a half chord perpendicular to the diameter $AB$ of the semicircle with center $O$. A circle with center $P$ is inscribed as shown in Figure 8.13, touching $AB$ at $E$ and arc $BD$ at $F$.Prove that $\Delta AED$ is isoceles.

*Solution.* The key observation is that $O, P$ and $F$ are collinear. This is because the circle $P$ is tangent to semicircle at $F$. Let $r$ be the radius of circle $P$ and $R$ be that of the semicircle. Then $OP = R - r$ and from the right triangle $OPE$,we get

$$(R - r)^2 = r^2 + (r + OC)^2$$

Also, $R^2 = OC^2 + CD^2$ from the right triangle $OCD$.Combining these two equations we get:

$$DC^2 = r^2 + 2rOC + 2rR$$

But

$$AE = R + r + OC$$

Therefore,

$$AE^2 = r^2 + 2r(R + OC) + (R + OC)^2 = CD^2 + AC^2$$

Since $AD^2 = CD^2 + AC^2$, $AE = AD$. □

**Problem. 8.1.16** Find the length of a side of an equilateral triangle in which the distances from its vertices to an interior point are $5, 7$, and $8$.

*Solution.* We have

$$\cos\theta = \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \frac{1}{2}$$

We also have

$$\begin{aligned} \cos(60^\circ + \theta) = \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} &= \frac{5^2 + 8^2 - s^2}{2 \cdot 5 \cdot 8} \\ \implies s^2 &= 129 \end{aligned}$$

□

## §8.2 Complex Numbers in Geometry

**Problem. 8.4.5** Let $A_0, A_1, A_2, A_3, A_4$ divide a unit circle into five equal parts.Prove that the chords $A_0A_1$, $A_0A_2$ satisfy

$$(A_0A_1 \cdot A_0A_2)^2 = 5$$

*Solution.* If $A_0 = 1$,then $A_1 = e^{i2\pi/5}$ and $A_2 = e^{-i2\pi/5}$.We have

$$\begin{aligned}
(A_0A_1 \cdot A_0A_2)^2 &= (e^{i2\pi/5} - 1)(e^{-i2\pi/5} - 1)(e^{-i2\pi/5} - e^{i2\pi/5})(e^{i2\pi/5} - e^{-i2\pi/5}) \\
&= 4(1 - \cos(2\pi/5))(1 - \cos(4\pi/5)) \\
&= 8(1 - \cos^2(\pi/5))(1 + \cos(\pi/5))
\end{aligned}$$

When $5\theta = \pi$, we have

$$\begin{aligned}
\cos 3\theta &= -\cos 2\theta \\
\implies 4\cos^3\theta - 3\cos\theta &= -2\cos^2\theta + 1 \\
\implies 4\cos^3\theta + 2\cos^2\theta - 3\cos\theta - 1 &= 0
\end{aligned}$$

$$\begin{aligned}
\cos 4\theta &= -\cos\theta \\
\implies 8\cos^4\theta - 8\cos^2\theta + 1 &= -\cos\theta \\
\implies 8\cos^2\theta(\cos^2\theta - 1) &= -(1 + \cos\theta) \\
\implies 8\cos^2\theta(\cos\theta - 1) &= -1 \\
\implies 8\cos^3\theta - 8\cos^2\theta + 1 &= 0
\end{aligned}$$

From the above two equations, we have

$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

Therefore, $\cos\theta = \frac{\sqrt{5}+1}{4}$. Therefore,

$$(A_0A_1 \cdot A_0A_2)^2 = 8(1 - \cos^2(\pi/5))(1 + \cos(\pi/5)) = 8(\frac{10 - 2\sqrt{5}}{16})(\frac{\sqrt{5} + 5}{4}) = 5$$

□

**Problem. 8.4.6** Given a point on the circumference of a unit circle and the vertices $A_1, A_2, \dots, A_n$ of an inscribed regular polygon of $n$ sides,prove that $PA_1^4 + PA_2^4 + \dots + PA_n^4$ is a constant (i.e., independent of the position of $P$ on the circumference).

*Solution.* The vertices of the regular polygon can be represented by $e^{i2\pi k/n}$ where $k = 0, 1, 2, \dots, n-1$. Let $z$ be any point on the circumference of the circle. We have

$$\begin{aligned}
PA_1^4 + PA_2^4 + \dots + PA_n^4 &= \sum_{k=0}^{n-1} \left((z - e^{\frac{i2\pi k}{n}})(\bar{z} - e^{-\frac{i2\pi k}{n}}\right)^2 \\
&= \sum_{k=0}^{n-1} \left(2 - ze^{-\frac{i2\pi k}{n}} - \bar{z}e^{\frac{i2\pi k}{n}}\right)^2 \\
&= \sum_{k=0}^{n-1} 4 + z^2e^{-\frac{i4\pi k}{n}} + \bar{z}^2e^{\frac{i4\pi k}{n}} - 4ze^{-\frac{i2\pi k}{n}} - 4\bar{z}e^{\frac{i2\pi k}{n}} + 2 \\
&= 6n
\end{aligned}$$

□

**Problem. 8.4.7** Let G denote the centroid of triangle ABC.Prove that

$$3(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2$$

*Solution.* WLOG, we can assume that the centroid of the triangle with vertices at $A(z_1)$,$B(z_2)$ and $C(z_3)$ is located at 0. We then have $z_1 + z_2 + z + 3 = 0$.

$$\begin{aligned}
AB^2 + BC^2 + CA^2 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z_2 - z_3)(\bar{z}_2 - \bar{z}_3) + (z_3 - z_1)(\bar{z}_3 - \bar{z}_1) \\
&= z_1\bar{z}_1 + z_2\bar{z}_2 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2 + z_3\bar{z}_3 - z_2\bar{z}_3 - z_3\bar{z}_2 + z_3\bar{z}_3 + z_1\bar{z}_1 - z_1\bar{z}_3 - z_3\bar{z}_1 \\
&= 2(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3) - \bar{z}_1(z_2 + z_3) - \bar{z}_2(z_1 + z_3) - \bar{z}_3(z_1 + z_2) \\
&= 2(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3) - \bar{z}_1(-z_1) - \bar{z}_2(-z_2) - \bar{z}_3(-z_3) \\
&= 3(z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3) = 3(GA^2 + GB^2 + GC^2)
\end{aligned}$$

□

**Problem. 8.4.8** Let ABCDEF be a hexagon in a circle of radius r. Show that if $AB = CD = EF = r$, then the midpoints of BC, DE and FA are the vertices of an equilateral triangle.

*Solution.* If O is the center of the circle circumscribing the hexagon, OAB, OCD and OEF are equilateral triangles. If A, C, and E are represented by $z_1, z_2$ and $z_3$, the coordinates of B, D and F are $z_1 e^{i\pi/3}$, $z_2 e^{i\pi/3}$ and $z_3 e^{i\pi/3}$.The midpoints of BC, DE and FA are $(z_1 e^{i\pi/3} + z_2)/2$, $(z_2 e^{i\pi/3} + z_3)/2$ and $(z_3 e^{i\pi/3} + z_1)/2$.

Whenever $c_1, c_2$ and $c_3$ are complex numbers that are the vertices of an equilateral triangle, we have

$$c_1^2 + c_2^2 + c_3^2 = c_1c_2 + c_2c_3 + c_3c_1$$

We have,

$$\begin{aligned}
&\left(\frac{z_1 e^{i\pi/3} + z_2}{2}\right)^2 + \left(\frac{z_2 e^{i\pi/3} + z_3}{2}\right)^2 + \left(\frac{z_3 e^{i\pi/3} + z_1}{2}\right)^2 \\
&= \frac{z_1^2 e^{i2\pi/3} + z_2^2 + 2z_1z_2 e^{i\pi/3}}{4} + \frac{z_2^2 e^{i2\pi/3} + z_3^2 + 2z_1z_3 e^{i\pi/3}}{4} + \frac{z_3^2 e^{i2\pi/3} + z_1^2 + 2z_1z_3 e^{i\pi/3}}{4} \\
&= \frac{z_1^2(e^{i2\pi/3} + 1) + z_2^2(e^{i2\pi/3} + 1) + z_3^2(e^{i2\pi/3} + 1) + 2z_1z_2 e^{i\pi/3} + 2z_2z_3 e^{i\pi/3} + 2z_3z_1 e^{i\pi/3}}{4} \\
&= \frac{z_1^2 e^{i\pi/3} + z_2^2 e^{i\pi/3} + z_3^2 e^{i\pi/3} + 2z_1z_2 e^{i\pi/3} + 2z_2z_3 e^{i\pi/3} + 2z_3z_1 e^{i\pi/3}}{4}
\end{aligned}$$

We also have,

$$\begin{aligned}
&\left(\frac{z_1 e^{i\pi/3} + z_2}{2}\right)\left(\frac{z_2 e^{i\pi/3} + z_3}{2}\right) + \left(\frac{z_2 e^{i\pi/3} + z_3}{2}\right)\left(\frac{z_3 e^{i\pi/3} + z_1}{2}\right) + \left(\frac{z_3 e^{i\pi/3} + z_1}{2}\right)\left(\frac{z_1 e^{i\pi/3} + z_2}{2}\right) \\
&= \frac{z_1^2 e^{i\pi/3} + z_2^2 e^{i\pi/3} + z_3^2 e^{i\pi/3} + z_1z_2(1 + e^{i\pi/3} + e^{i2\pi/3}) + z_2z_3(1 + e^{i\pi/3} + e^{i2\pi/3}) + z_3z_1(1 + e^{i\pi/3} + e^{i2\pi/3})}{4} \\
&= \frac{z_1^2 e^{i\pi/3} + z_2^2 e^{i\pi/3} + z_3^2 e^{i\pi/3} + 2z_1z_2 e^{i\pi/3} + 2z_2z_3 e^{i\pi/3} + 2z_3z_1 e^{i\pi/3}}{4}
\end{aligned}$$

Therefore, the midpoints of BC, DE and FA form an equilateral triangle. □

**Problem. 8.4.9** If $z_1, z_2, z_3$ are such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$,show that $z_1, z_2, z_3$ are the vertices of an equilateral triangle inscribed in a unit circle.

*Solution.* The lengths of the three medians of the triangle are given by

$$\begin{aligned}\left|\frac{z_1+z_2-2z_3}{2}\right| &= \left|\frac{-z_3-2z_3}{2}\right| = \frac{3}{2}\\ \left|\frac{z_1+z_3-2z_2}{2}\right| &= \left|\frac{-z_2-2z_2}{2}\right| = \frac{3}{2}\\ \left|\frac{z_2+z_3-2z_1}{2}\right| &= \left|\frac{-z_1-2z_1}{2}\right| = \frac{3}{2}\end{aligned}$$

As the lengths of the three medians are equal, the triangle whose vertices are $z_1, z_2$ and $z_3$ is an equilateral triangle. □

**Problem. 8.4.10** Show that $z_1, z_2, z_3$ form an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

*Solution.* Rotating one side by 60° we get the second side in an equilateral triangle, so we have

$$\begin{aligned}z_3 - z_1 &= (z_2 - z_1)e^{i\pi/3}\\ \iff \frac{z_3-z_1}{z_2-z_1} - \frac{1}{2} &= i\frac{\sqrt{3}}{2}\\ \iff \frac{(2z_3-z_1-z_2)^2}{4(z_2-z_1)^2} &= \frac{-3}{4}\\ \iff 4z_3^2 + z_1^2 + z_2^2 + 2z_1z_2 - 4z_3z_1 - 4z_3z_2 &= -3z_2^2 - 3z_1^2 + 6z_2z_1\\ \iff 4z_3^2 + 4z_1^2 + 4z_2^2 &= 4z_1z_2 + 4z_2z_3 + 4z_3z_1\\ \iff z_1^2 + z_2^2 + z_3^2 &= z_1z_2 + z_2z_3 + z_3z_1\end{aligned}$$

□

**Problem. 8.4.11** The three points in the complex plane which correspond to the roots of the equation

$$z^3 - 3pz^2 + 3qz - r = 0$$

are the vertices of a triangle.

(a) Prove that the centroid of the triangle is the point corresponding to $p$.

(b) Prove that $ABC$ is an equilateral triangle if and only if $p^2 = q$.

*Solution.* If $z_1, z_2, z_3$ are the roots of the above equation, we have

$$\begin{aligned}z_1 + z_2 + z_3 &= 3p\\ z_1z_2 + z_2z_3 + z_3z_1 &= 3q\end{aligned}$$

(a) If $z_1, z_2, z_3$ are the vertices of a triangle,we have the centroid at $(z_1 + z_2 + z_3)/3 = p$.

(b) If $z_1, z_2, z_3$ are the vertices of an equilateral triangle, we have

$$\begin{aligned}z_1^2 + z_2^2 + z_3^2 &= z_1z_2 + z_2z_3 + z_3z_1\\ \implies 9p^2 - 6q &= 3q\\ \implies p^2 &= q\end{aligned}$$

□



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